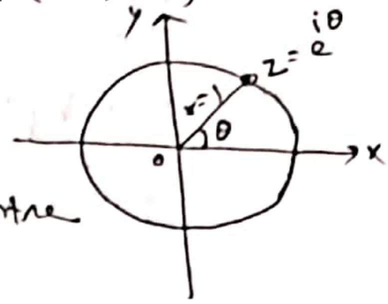
TDC part III Mathematical physics paper-V

Ques:- Evaluate the following:

(i) $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$ (ii) $\int_0^{2\pi} \frac{d\theta}{2+\cos\theta}$ (iii) $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta} d\theta$ (iv) $\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta} d\theta$.

Rule: Integration round unit circle of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$ where $f(\cos\theta, \sin\theta)$ is a rotational function of $\cos\theta$ and $\sin\theta$.



* Convert $\cos\theta$ and $\sin\theta$ into z.

Consider a circle of unit radius with centre at origin as a contour,

$$z = re^{i\theta} = 1 . e^{i\theta} \Rightarrow z = \cos\theta + i\sin\theta \qquad \because r = 1$$

$$z^{-1} = \frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} \Rightarrow z^{-1} = \cos\theta - i\sin\theta$$

on adding, $z + z^{-1} = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2}(z + \frac{1}{z})$

on subtracting, $z - z^{-1} = 2i\sin\theta \Rightarrow \sin\theta = \frac{1}{2i}(z - \frac{1}{z})$

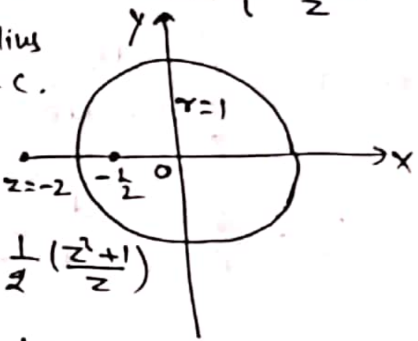
$$z^2 = e^{i2\theta} = \cos 2\theta + i\sin 2\theta$$

$$z^{-2} = \frac{1}{z^2} = \cos 2\theta - i\sin 2\theta \Rightarrow z^2 + z^{-2} = 2\cos 2\theta \Rightarrow \cos 2\theta = \frac{1}{2}(z^2 + \frac{1}{z^2})$$

and $\sin 2\theta = \frac{1}{2i}(z^2 - \frac{1}{z^2})$.

Again $z = e^{i\theta} \Rightarrow dz = e^{i\theta} . i d\theta \Rightarrow dz = iz d\theta \Rightarrow d\theta = \frac{1}{i}\frac{dz}{z}$.

*(i) $I = \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta}$, consider a circle of unit radius with centre at origin as a contour c.



$$z = e^{i\theta} \Rightarrow dz = e^{i\theta} . i d\theta = iz d\theta \Rightarrow d\theta = \frac{1}{i} . \frac{dz}{z}$$

$$z = e^{i\theta} = \cos\theta + i\sin\theta$$

$$z^{-1} = \frac{1}{z} = \cos\theta - i\sin\theta \Rightarrow 2\cos\theta = z + \frac{1}{z} \Rightarrow \cos\theta = \frac{1}{2}\left(\frac{z^2+1}{z}\right)$$

Now $I = \int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \int_c \frac{\frac{1}{i} . \frac{dz}{z}}{5 + 4 . \frac{1}{2} . \frac{z^2+1}{z}} = \frac{1}{i}\int_c \frac{dz}{5z + 2z^2 + 2}$

$$= \frac{1}{i}\int_c \frac{dz}{2z^2+5z+2} = \frac{1}{i}\int_c \frac{dz}{2z^2+4z+z+2} = \frac{1}{i}\int \frac{dz}{2z(z+2)+1(z+2)}$$

$$\Rightarrow I = \frac{1}{i}\int_c \frac{dz}{(z+2)(2z+1)} = \frac{1}{i}\int_c f(z)\, dz$$

where $f(z) = \frac{1}{(z+2)(2z+1)}$

for poles of $f(z)$, $(z+2)(2z+1) = 0 \Rightarrow z = -2, -\frac{1}{2}$.

Therefore, there is only pole $z = -\frac{1}{2}$ lies inside the unit circle c.

so $\underset{z=-\frac{1}{2}}{\text{Res}}\, f(z) = \lim_{z\to -\frac{1}{2}} (z+\frac{1}{2}) f(z) = \lim_{z\to-\frac{1}{2}} \frac{(2z+1)}{2} \cdot \frac{1}{(2z+1)(z+2)}$

$= \lim_{z\to-\frac{1}{2}} \frac{1}{2(z+2)} = \frac{1}{2(-\frac{1}{2}+2)} = \frac{1}{3}.$

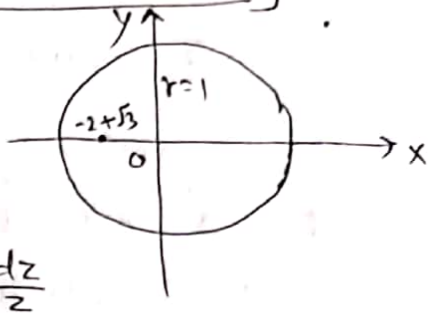
$\underset{z=-\frac{1}{2}}{\text{Res}}\, f(z) = \frac{1}{3}$

Using Cauchy's Residue theorem

$\int_c f(z)\,dz = 2\pi i \times \underset{z=-\frac{1}{2}}{\text{Res}}\, f(z) = 2\pi i \times \frac{1}{3} = \frac{2\pi i}{3}$

Now $I = \frac{1}{i}\int_c f(z)\,dz = \frac{1}{i} \times \frac{2\pi i}{3} \Rightarrow \boxed{\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} = \frac{2\pi}{3}}$

(ii) $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta}.$



consider a circle of unit radius with centre at origin as a closed contour c.

$z = e^{i\theta} \Rightarrow dz = e^{i\theta}.\, i\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{i}\frac{dz}{z}$

$z = e^{i\theta} = \cos\theta + i\sin\theta$
$z^{-1} = \frac{1}{z} = \cos\theta - i\sin\theta$ $\Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$

Now $I = \int_0^{2\pi} \frac{d\theta}{2+\cos\theta} = \int_c \frac{\frac{1}{i}\frac{dz}{z}}{2+\frac{z^2+1}{2z}} = \frac{1}{i}\int_c \frac{2\,dz}{z^2+4z+1}$

$I = \frac{2}{i}\int_c \frac{dz}{z^2+4z+1} = \frac{2}{i}\int_c f(z)\,dz$

where $f(z) = \frac{1}{z^2+4z+1}.$

For poles of $f(z)$, $z^2+4z+1=0 \Rightarrow z = \frac{-4\pm\sqrt{4^2-4\cdot1\cdot1}}{2\cdot1} = \frac{-4\pm2\sqrt{3}}{2}$

$\Rightarrow z = -2\pm\sqrt{3} \Rightarrow z = -2+\sqrt{3}$ or $z = -2-\sqrt{3}.$

Therefore, there is only one pole $z=-2+\sqrt{3}$ lies inside the unit circle

so $\underset{z=-2+\sqrt{3}}{\text{Res}}\, f(z) = \lim_{z\to-2+\sqrt{3}} (z+2-\sqrt{3}) f(z) = \lim_{z\to-2+\sqrt{3}} (z+2-\sqrt{3}) \frac{1}{(z+2-\sqrt{3})(z+2+\sqrt{3})}$

$= \lim_{z\to-2+\sqrt{3}} \frac{1}{z+2+\sqrt{3}} = \frac{1}{-2+\sqrt{3}+2+\sqrt{3}} = \frac{1}{2\sqrt{3}}.$

$\underset{z=-2+\sqrt{3}}{\text{Res}}\, f(z) = \frac{1}{2\sqrt{3}}.$

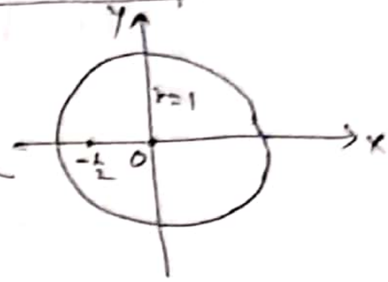
using Cauchy's Residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \operatorname*{Res}_{z=-2+\sqrt{3}} f(z) = 2\pi i \times \frac{1}{2\sqrt{3}}$$

$$\Rightarrow \int_C f(z)\,dz = \frac{\pi i}{\sqrt{3}}$$

Now $I = \frac{2}{i}\int_C f(z)\,dz = \frac{2}{i}\times\frac{\pi i}{\sqrt{3}} \Rightarrow \boxed{\int_0^{2\pi}\frac{d\theta}{2+\cos\theta} = \frac{2\pi}{\sqrt{3}}}$

(ii) $I = \int_0^{2\pi}\frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$



Consider a circle of unit radius with centre at origin as a closed contour C.

$z = e^{i\theta} \Rightarrow dz = e^{i\theta}\cdot i\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{i}\frac{dz}{z}$.

$$\left.\begin{aligned} z &= e^{i\theta} = \cos\theta + i\sin\theta \\ z^{-1} &= \frac{1}{z} = \cos\theta - i\sin\theta \end{aligned}\right\} \Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \Rightarrow \cos\theta = \frac{z^2+1}{2z}.$$

Again
$$\left.\begin{aligned} z^2 &= e^{i2\theta} = \cos 2\theta + i\sin 2\theta \\ z^{-2} &= \frac{1}{z^2} = \cos 2\theta - i\sin 2\theta \end{aligned}\right\} \Rightarrow \cos 2\theta = \frac{1}{2}\left(z^2+\frac{1}{z^2}\right) \Rightarrow \cos 2\theta = \frac{z^4+1}{2z^2}$$

Now
$$I = \int_C \frac{\frac{z^4+1}{2z^2}}{5+4\cdot\frac{z^2+1}{2z}}\cdot\frac{1}{i}\frac{dz}{z} = \frac{1}{2i}\int_C \frac{z^4+1}{z^2(5z+2z^2+2)}\,dz$$

$$\Rightarrow I = \frac{1}{2i}\int_C \frac{z^4+1}{z^2(2z+1)(z+2)}\,dz = \frac{1}{2i}\int_C f(z)\,dz$$

when $f(z) = \frac{z^4+1}{z^2(2z+1)(z+2)}$

For poles of $f(z)$, $z^2(2z+1)(z+2) = 0$

$$\Rightarrow z = 0, 0, -\tfrac{1}{2}, -2$$

Therefore $f(z)$ has double pole at $z=0$ and single pole at $z=-\frac{1}{2}$ inside the unit circle.

$$\operatorname*{Res}_{z=0} f(z) = \lim_{z\to 0}\frac{1}{\lfloor 2-1}\frac{d}{dz}z^2 f(z) = \lim_{z\to 0}\frac{1}{\lfloor 1}\frac{d}{dz}z^2\cdot\frac{z^4+1}{z^2(2z+1)(z+2)}$$

$$= \lim_{z\to 0}\frac{d}{dz}\left(\frac{z^4+1}{2z^2+5z+2}\right) = \lim_{z\to 0}\frac{(2z^2+5z+2)\cdot 4z^3 - (z^4+1)\cdot(4z+5)}{(2z^2+5z+2)^2}$$

$$= \frac{2\cdot 0 - 1\cdot 5}{2^2} = -\frac{5}{4} \Rightarrow \operatorname*{Res}_{z=0} f(z) = -\frac{5}{4}$$

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and $\underset{z=-\frac{1}{2}}{\operatorname{Res}} f(z) = \lim_{z\to -\frac{1}{2}} (z+\frac{1}{2}) f(z) = \lim_{z\to -\frac{1}{2}} \frac{(2z+1)}{2} \cdot \frac{z^4+1}{z^2(2z+1)(z+2)}$

$$= \lim_{z\to -\frac{1}{2}} \frac{z^4+1}{2z^2(z+2)} = \frac{\frac{1}{16}+1}{2\cdot\frac{1}{4}(-\frac{1}{2}+2)} = \frac{17}{16\times\frac{3}{4}} = \frac{17}{12}$$

$$\underset{z=-\frac{1}{2}}{\operatorname{Res}} f(z) = \frac{17}{12}$$

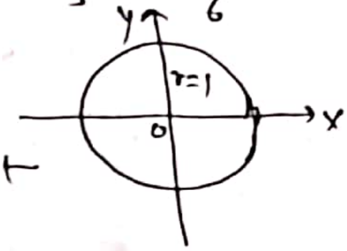
Using Cauchy's Residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \left\{ \underset{z=0}{\operatorname{Res}} f(z) + \underset{z=-\frac{1}{2}}{\operatorname{Res}} f(z) \right\} = 2\pi i \left(-\frac{5}{4} + \frac{17}{12}\right)$$

$$= 2\pi i \,\frac{2}{12} \Rightarrow \int_C f(z)\,dz = \frac{\pi i}{3}$$

Now $I = \frac{1}{2i}\int_C f(z)\,dz \Rightarrow \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \frac{1}{2i}\times\frac{\pi i}{3} = \frac{\pi}{6}$

(iv) $I = \int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\,d\theta$



Consider a circle of unit radius with centre at origin as a closed contour C.

$z = e^{i\theta} \Rightarrow dz = e^{i\theta}.\,i\,d\theta = iz\,d\theta \Rightarrow d\theta = \frac{1}{i}\frac{dz}{z}$.

$\left.\begin{array}{l} z = e^{i\theta} = \cos\theta + i\sin\theta \\ z^{-1} = \frac{1}{z} = \cos\theta - i\sin\theta \end{array}\right\} \Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) = \frac{z^2+1}{2z}$.

Now $I = \int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\,d\theta = \int_C \frac{\frac{z^2+1}{2z}}{5+4.\frac{z^2+1}{2z}} \cdot \frac{1}{i}\frac{dz}{z} = \frac{1}{2i}\int \frac{(z^2+1)\,dz}{z(5z+2z^2+2)}$

$$I = \frac{1}{2i}\int_C \frac{(z^2+1)\,dz}{z(2z+1)(z+2)} = \frac{1}{2i}\int_C f(z)\,dz$$

where $f(z) = \frac{z^2+1}{z(2z+1)(z+2)}$

For poles of $f(z)$, $z\cdot(2z+1)(z+2) = 0 \Rightarrow z = 0, -\frac{1}{2}, -2$

Therefore $f(z)$ has two single poles at $z=0$, $z=-\frac{1}{2}$ inside the unit circle.

$\underset{z=0}{\operatorname{Res}} f(z) = \lim_{z\to 0} z.f(z) = \lim_{z\to 0} z.\frac{z^2+1}{z(2z+1)(z+2)} = \lim_{z\to 0}\frac{z^2+1}{(2z+1)(z+2)} = \frac{1}{1\cdot 2} = \frac{1}{2}$.

$\underset{z=-\frac{1}{2}}{\operatorname{Res}} f(z) = \lim_{z\to-\frac{1}{2}}(z+\frac{1}{2})f(z) = \lim_{z\to-\frac{1}{2}} \frac{2z+1}{2}\cdot\frac{z^2+1}{z(2z+1)(z+2)} = \lim_{z\to-\frac{1}{2}}\frac{z^2+1}{2z(z+2)} = \frac{\frac{1}{4}+1}{-1.(-\frac{1}{2}+2)}$

$$= \frac{5}{4}\times -\frac{2}{3} = -\frac{5}{6}.$$

Using Cauchy's Residue theorem.

$$\int_C f(z)\,dz = 2\pi i \times \left\{ \underset{z=0}{\operatorname{Res}} f(z) + \underset{z=-\frac{1}{2}}{\operatorname{Res}} f(z) \right\} = 2\pi i \times \left(\frac{1}{2} - \frac{5}{6}\right) = 2\pi i \times -\frac{1}{3} = -\frac{2\pi i}{3}$$

Now $I = \frac{1}{2i}\oint_C f(z)\,dz = \frac{1}{2i}\cdot\left(-\frac{2\pi i}{3}\right)$

$$\Rightarrow \int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\,d\theta = -\frac{\pi}{3}.$$

Ques:- State and prove Cauchy's Residue theorem.

or, If a function $f(z)$ is analytic on and inside a closed contour C except at finite number of poles then prove that

$$\oint_C f(z)\,dz = 2\pi i \sum R$$

where $\sum R$ = sum of residues at poles.

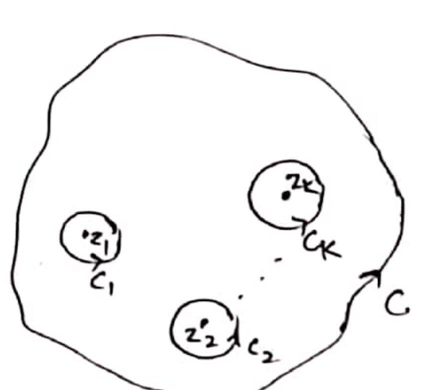
Ans:- Cauchy's Residue theorem: If a function $f(z)$ is analytic inside and on a simple closed contour C except at finite number of points $z_1, z_2, \ldots z_K$ interior to C and if $R_1, R_2, \ldots, R_K$ are residues of the function $f(z)$ at the isolated singular points (poles) $z_1, z_2, \ldots, z_K$ respectively then from Cauchy's residues theorem,

$$\oint_C f(z)\,dz = 2\pi i\,(R_1 + R_2 + \cdots + R_K) = 2\pi i \sum R$$

or $$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{K} \underset{z=z_k}{\mathrm{Res}}\, f(z)$$

Here integration should be taken counter clockwise along the contour C.

proof:- Suppose $c_1, c_2, \ldots c_K$ are circles around isolated singular points (poles) $z_1, z_2, \ldots z_K$ respectively within the simple closed contour C. All circles are disjoint i.e, no interior points of the circles are common. So the region is multiply connected domain.



Using Cauchy's integral theorem,

$$\oint_C f(z)\,dz = \oint_{c_1} f(z)\,dz + \oint_{c_2} f(z)\,dz + \cdots + \oint_{c_K} f(z)\,dz \quad \text{———} (1)$$

Since $\oint_C f(z)dz = 2\pi i \cdot \operatorname{Res}_{z=z_0} f(z)$

so $\oint_{C_1} f(z)dz = 2\pi i \cdot \operatorname{Res}_{z=z_1} f(z)$, $\oint_{C_2} f(z)dz = 2\pi i \cdot \operatorname{Res}_{z=z_2} f(z)$

. . . . . and $\oint_{C_K} f(z)dz = 2\pi i \operatorname{Res}_{z=z_K} f(z)$

using these in eqn (1), we get

$$\oint_C f(z)dz = 2\pi i \cdot \operatorname{Res}_{z=z_1} f(z) + 2\pi i \cdot \operatorname{Res}_{z=z_2} f(z) + \dots + 2\pi i \operatorname{Res}_{z=z_K} f(z)$$

$$= 2\pi i \times \left[\operatorname{Res}_{z=z_1} f(z) + \operatorname{Res}_{z=z_2} f(z) + \dots + \operatorname{Res}_{z=z_K} f(z) \right]$$

$$\oint_C f(z)dz = 2\pi i \cdot \sum_{K=1}^{K} \operatorname{Res}_{z=z_K} f(z)$$ proved.